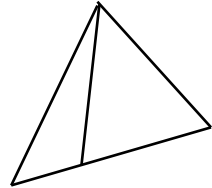


4. SPECIAL POINTS OF TRIANGLES

§4.1. Ceva's Theorem

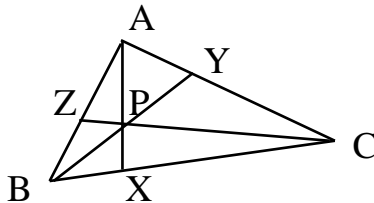
A **cevian** of a triangle is a line segment from a vertex to any point on the opposite side.



Suppose we have a triangle and three cevians, one from each vertex. We are interested in the cases where these three cevians are concurrent. Ceva's theorem gives a necessary and sufficient condition for this to occur. It is an interesting theorem since, although the result is about length not areas, it uses areas in the proof.

Theorem 1: Cevians AX , BY , CZ of $\triangle ABC$ are concurrent if and only if $\left(\frac{AZ}{ZB}\right)\left(\frac{BX}{XC}\right)\left(\frac{CY}{YA}\right) = 1$.

Proof: Suppose the cevians are concurrent at P .



$$\left(\frac{AZ}{ZB}\right) = \left(\frac{\triangle CAZ}{\triangle CZB}\right) = \left(\frac{\triangle PAZ}{\triangle PZB}\right) = \left(\frac{\triangle CAZ - \triangle PAZ}{\triangle CZB - \triangle PZB}\right)$$

$$= \left(\frac{\Delta PAC}{\Delta PBC} \right).$$

Similarly, $\left(\frac{BX}{XC} \right) = \left(\frac{\Delta PAB}{\Delta PAC} \right)$ and $\left(\frac{CY}{YA} \right) = \left(\frac{\Delta PBC}{\Delta PAB} \right)$.

$$\text{So } \left(\frac{AZ}{ZB} \right) \left(\frac{BX}{XC} \right) \left(\frac{CY}{YA} \right) = \left(\frac{\Delta PAC}{\Delta PBC} \right) \left(\frac{\Delta PAB}{\Delta PAC} \right) \left(\frac{\Delta PBC}{\Delta PAB} \right) = 1.$$

Now suppose that $\left(\frac{AZ}{ZB} \right) \left(\frac{BX}{XC} \right) \left(\frac{CY}{YA} \right) = 1$.

Suppose that $AX \cap BY = P$ and let $Z' = CP \cap AB$.

Hence by the first part of the proof:

$$\left(\frac{AZ'}{Z'B} \right) \left(\frac{BX}{XC} \right) \left(\frac{CY}{YA} \right) = 1 \text{ and so}$$

$$\left(\frac{AZ}{ZB} \right) = \left(\frac{AZ'}{Z'B} \right) \text{ which means that } Z = Z'.$$

§4.2. Special Points In A Triangle

Suppose we have some property of a cevian in relation to the triangle. Suppose that in a triangle ABC we have cevians from each of the three vertices, all having these properties. In many cases these three cevians are concurrent. We call such a property a **special property** and the point of concurrence we call a **special point** of the triangle. These special points have some interesting properties.

The most well-known special property is that of the cevian bisecting the opposite side. The cevians with

this property are called **medians** and the corresponding special point is called the **centroid**.

Theorem 2: The medians of a triangle are concurrent at a point that is $2/3$ of the way down each median, from the vertex.

Proof: Let the vertices A, B, C be represented by the vectors \mathbf{a} , \mathbf{b} , \mathbf{c} respectively. The mid-point of BC will be represented by $\frac{\mathbf{b} + \mathbf{c}}{2}$. The point that is $2/3$ of the way from A to this mid-point is:

$$1/3\mathbf{a} + 2/3 \left(\frac{\mathbf{b} + \mathbf{c}}{2} \right) = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}.$$

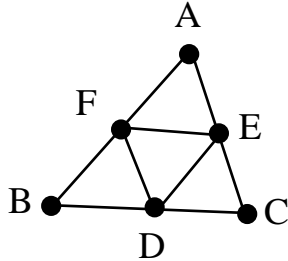
By symmetry this will also be the point that lies $2/3$ of the way down the other medians. Hence the medians are concurrent at the point represented by $\frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$.

The centroid of a triangle has a physical interpretation in that if the triangle is made of some uniform material, the centroid is the centre of gravity.

Theorem 3: Let the midpoints of $\mathcal{P}(ABC)$ be D, E, F. The edges of $\mathcal{P}(DEF)$ split $\mathcal{P}(ABC)$ into four triangles, all of them congruent to one another, and all similar to $\mathcal{P}(ABC)$. Moreover $\mathcal{P}(ADEF)$, $\mathcal{P}(BDFE)$ and $\mathcal{P}(DECF)$ are parallelograms.

Proof: Consider the points as vectors with the corresponding letters.

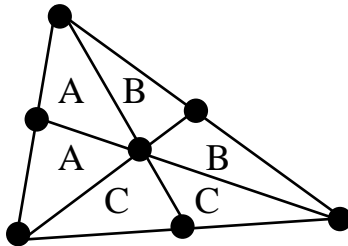
Then $\mathbf{d} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\mathbf{e} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$, $\mathbf{f} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.
 FE is represented by $\frac{1}{2}(\mathbf{a} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(\mathbf{c} - \mathbf{b})$.
 BC is represented by $\mathbf{c} - \mathbf{b}$.



Hence $FE \parallel BC$ and $|FE| = \frac{1}{2} |BC|$.
 Similarly $DF \parallel CA$ and $|DF| = \frac{1}{2} |AC|$ and
 $ED \parallel AB$ and $|ED| = \frac{1}{2} |AB|$.
 Hence $|AF| = |FB| = \frac{1}{2} |AB| = |DE|$,
 $|AE| = |EC| = \frac{1}{2} |AC| = |DF|$ and
 $|BD| = |DC| = \frac{1}{2} |BC| = |FE|$.
 The rest is easily checked.

Theorem 4: The medians of a triangle divide it into 6 smaller triangles of equal area to one another.

Proof:



Since triangles with the same vertex the sub-triangles with the same label have equal area. But the larger

triangles bounded by a side, half a side and a median, also have equal area so:

$$2A + C = 2B + C;$$

$$2A + B = 2C + B \text{ and}$$

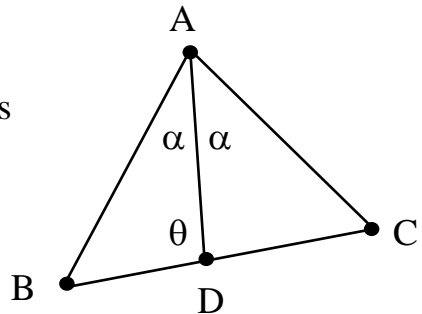
$$2B + A = 2C + A$$

from which we get $A = B = C$.

Another special property of a cevian is that of bisecting the corresponding angle. The special point in this case is called the **incentre**, because it is the centre of the circle that touches all three sides of the triangle. But, before we prove this, here is a result about one such bisecting cevian.

Theorem 5: If AD is the cevian of $\triangle ABC$ which bisects

$$\angle BAC \text{ then } \frac{BD}{CD} = \frac{AB}{AC}.$$



Proof: Suppose AD bisects $\angle BAC$ and let $\angle BAD = \angle DAC = \alpha$. Let $\angle ADB = \theta$. By the extended sine rule in the two smaller triangles:

$$\frac{BD}{\sin \alpha} = \frac{AB}{\sin \theta} \text{ and } \frac{CD}{\sin \alpha} = \frac{AC}{\sin(180 - \theta)} = \frac{AC}{\sin \theta}.$$

$$\text{Hence } \frac{BD}{CD} = \left(\frac{AB \sin \alpha}{\sin \theta} \right) \left(\frac{\sin \theta}{AC \sin \alpha} \right) = \frac{AB}{AC}.$$

Theorem 6: The bisectors of the three angles of a triangle are concurrent.

Proof: Let O be the intersection of the bisectors of the angles $\angle BAC$ and $\angle ACB$. Let P, Q, R be the feet of the perpendiculars from O to AB, AC and BC respectively.

$$\text{Then } \sin \frac{OP}{OA} = \sin \angle PAO = \sin \angle OAQ = \frac{OQ}{OA}.$$

Hence $OP = OQ$. Similarly $OQ = OR$, whence $OP = OR$.

$$\text{Then } \sin \angle ABO = \frac{OP}{OB} = \frac{OR}{OB} = \sin \angle OBC.$$

So OB bisects $\angle ABC$.

Corollary: The circle with centre O and radius

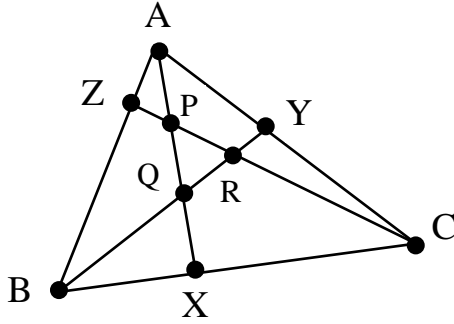
$$OP = OQ = OR$$

touches the three sides of the triangle.

A third special property is that of a cevian being perpendicular to the opposite side. These cevians are called **altitudes** and the corresponding special point is called the **orthocentre**.

Theorem 7: The altitudes of a triangle are concurrent.

Proof: Let the triangle be ABC and let the feet of the perpendiculars from A, B, C be X, Y, Z respectively. Suppose the altitudes are not concurrent and suppose that $P = AX \cap CZ$, $Q = AX \cap BY$ and $R = CZ \cap BY$.



$$\angle BAX = 90 - \angle APZ = 90 - \angle QPR = \angle ZCB.$$

Hence $\frac{BX}{AB} = \sin \angle BAX = \sin \angle ZCB = \frac{BZ}{BC}$ and hence

$$\frac{BX}{BZ} = \frac{AB}{BC}.$$

Similarly $\frac{CY}{CX} = \frac{BC}{CA}$ and $\frac{AZ}{AY} = \frac{CA}{AB}$.

(Rather go through the whole process twice more simply apply the replacements $A \rightarrow B \rightarrow C \rightarrow A$ and $X \rightarrow Y \rightarrow Z \rightarrow X$ twice.)

$$\begin{aligned} \text{Hence } \left(\frac{AZ}{ZB}\right) \left(\frac{BX}{XC}\right) \left(\frac{CY}{YA}\right) &= \left(\frac{AZ}{YA}\right) \left(\frac{BX}{ZB}\right) \left(\frac{CY}{XC}\right) \\ &= \left(\frac{CA}{AB}\right) \left(\frac{AB}{BC}\right) \left(\frac{BC}{CA}\right) = 1. \end{aligned}$$

So by Ceva's theorem the altitudes are concurrent.

We extend the term ‘special point’ to include the point of concurrence of the three perpendicular bisectors of the three sides. These are not Cevians because they don’t pass through the vertices, but they are concurrent.

Theorem 8: The perpendicular bisectors of the three sides of a triangle are concurrent.

Proof: This time we start with the point of concurrence. Since the vertices of the triangle are not collinear there exists a circle, \mathcal{C} , passing through them. Let C be the centre of this circle. Then AB , BC and CA are chords of \mathcal{C} and so their perpendicular bisectors pass through C .

The circle passing through the vertices is called the **circumcircle** and its centre is called the **circumcentre**.

To summarise here are the three special properties and the associated special point:

property of concurrent lines	point
Cevian bisecting side (median)	centroid
Cevian bisecting angle	incentre
Cevian perpendicular to side (altitude)	orthocentre
perpendicular bisector of side	circumcentre

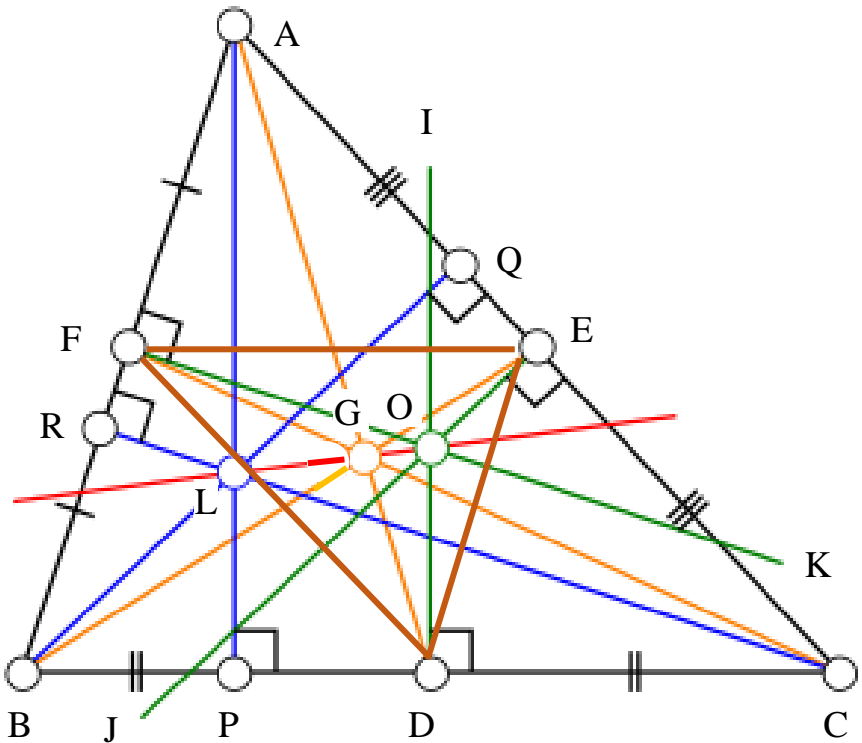
Three of these special points and the nine points at the ends of the cevians have some remarkable properties.

§4.3. The Euler Line

Theorem 9: The centroid, incentre and orthocentre of a triangle are collinear.

Proof: Consider the triangle $\mathcal{P}(ABC)$. Let AD , BE , CF be the medians. Now $\mathcal{P}(AFDE)$ is a parallelogram, so its diagonals AD and EF bisect each other.

Let AP , BQ , CR be the perpendiculars from A , B , C to BC , CA , AB respectively. Let L be the point of concurrence of these cevians, that is the orthocentre of $\mathcal{P}(ABC)$.



Let DI , EJ and FK be the perpendicular bisectors of $\mathcal{P}(ABC)$. Let O be the point of concurrence of these. It is the circumcentre of $\mathcal{P}(ABC)$, but also the orthocentre of $\mathcal{P}(DEF)$.

Now L is the orthocentre of $\mathcal{P}(ABC)$, while O is the orthocentre of the similar triangle $\mathcal{P}(DEF)$, $AL = 2OD$. But $AG = 2/3AD$, so $AG = 2GD$.

Since $AP \perp BC$ and $ID \perp BC$, $AP \parallel ID$.

$\therefore \angle PAG = \angle HDG$ and so $\mathcal{P}(PAG) \sim \mathcal{P}(HDG)$.

$\therefore \angle AGL = \angle DGH$ and hence G, L, O are collinear.

The line on which the centroid, orthocentre and the circumcentre all lie is called the **Euler line** of the triangle.

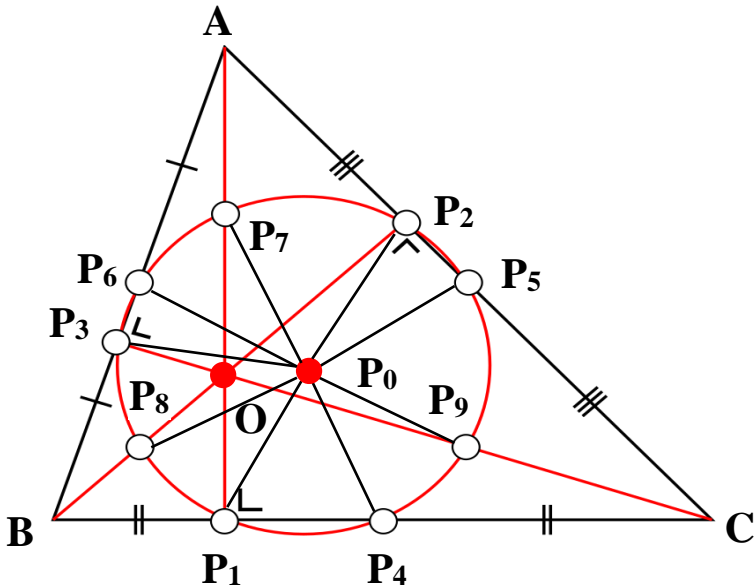
§4.4. The Nine Point Circle

With the three pairs of cevians we have so far discussed (medians, angle-bisectors and altitudes) we have nine points on any given triangle. The amazing thing is that all nine lie on a common circle, called the **9-point circle**.

Theorem 7: Let ABC be a triangle. Let P_1, P_2, P_3 be the feet of the perpendiculars from the vertices to the opposite sides. Let O be the orthocentre (the point where these perpendiculars meet – we have shown that they are concurrent).

Let P_4, P_5, P_6 be the midpoints of the three sides and let P_7, P_8, P_9 be the midpoints of AO, BO and CO respectively. Then $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9$ all lie on a common circle.

Proof: Since P_5 and P_6 are the midpoints of AC and AB respectively, then $P_5P_6 \parallel BC$ and $P_5P_6 = \frac{1}{2} BC$.



Hence $P_5P_6 \parallel P_8P_9$ and $P_5P_6 = P_8P_9$.

It follows that $P_5P_6P_8P_9$ is a parallelogram.

Since P_6 and P_8 are the midpoints of AB and OB respectively, then $P_6P_8 \parallel AO$ and $P_6P_8 = \frac{1}{2} AO$.

Now $P_6P_8 \parallel AO$ and $AO \perp BC$ and $P_5P_6 \parallel BC$, $P_6P_8 \perp P_5P_6$ and so $P_5P_6P_8P_9$ is a rectangle. Now P_5P_8 and P_6P_9 are the diagonals of this rectangle and suppose they

bisect each other at P_0 . Then $P_0P_5 = P_0P_6 = P_0P_7 = P_0P_8$. Similarly $P_4P_5P_7P_8$ and $P_4P_6P_7P_9$ are rectangles.

Since the second and third rectangles share a diagonal with the first, their vertices must lie on the same circle as the first rectangle.

So all 6 points, P_4, P_5, P_6, P_7, P_8 and P_9 lie on a common circle, with centre P_0 . Since $\angle P_4P_1P_7 = 90$, P_1 lies on this common circle. Similarly P_2 and P_3 lie in this circle. Hence all 9 points P_1, \dots, P_9 lie on a circle with centre P_0 .